Sharp Boundedness and Regularizing effects of the integral Menger curvature for submanifolds

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October 24, 2011

In this paper we show that embedded and compact C^1 manifolds have finite integral Menger curvature if and only if they are locally graphs of certain Sobolev-Slobodeckij spaces. Furthermore, we prove that for some intermediate energies of integral Menger type a similar characterization of objects with finite energy can be given.

Contents

1 Introduction

To study the geometry of metric spaces, Karl Menger found a way to define the curvature of a curve without using any parameterization of this geometric object [12]. For each triple of points (x, y, z) lying on the curve he looked at the reciprocal of the radius of the circumscribing circle of the three points. This quantity is nowadays named "Menger curvature" and will be denoted by $c(x, y, z)$ in this article. Menger observed that one obtains the curvature of the curve at a point p by the limit of the this Menger curvature (x, y, z) as the three points converge to p.

The growing interest in this quantity during the last years started with the observation that Menger curvature has a tight relation to many modern fields in mathematics apart from metric geometry. A milestone is certainly the discovery of the intimate relation between total Menger curvature of an $\mathcal{H}^1$ measurable set K - given by

$$\mathcal{M}_2(K) := \int_K \int_K \int_K c^2(x, y, z)\, d\mathcal{H}^1_x\, d\mathcal{H}^1_y\, d\mathcal{H}^1_z \quad -$$

and harmonic analysis, rectifiability, and analytic capacity (see [11] or [16]). M. Leger proved in [8] that finite global Menger curvature implies that the set is rectifiable. Using this result, Guy

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David proved that $\mathcal{M}_2(K) < \infty$ is a sufficient condition for a set K to have vanishing analytic capacity. This enabled him to prove the Vitushkin's conjecture [3] for sets of finite one-dimensional Hausdorff measure.

Another application of Menger curvature is its use as basic building block in the construction of so called "knot energies" - energies that penalize self intersections and thus can hopefully be minimized within a given knot class. These energies play an important role in the modeling of the structure of polymer chains like proteins and DNA.

The first to use Menger curvature to define such self-repulsive energies were Oscar Gonzales and John H. Maddocks. In [4], they introduced and analyzed the notion of *global radius of curvature* of a curve γ given by

$$\rho(\gamma) := \inf_{x,y,z\in\gamma(\mathbb{R}/\mathbb{Z})} \frac{1}{c(x,y,z)}.$$

At the end of this article, they also suggest the investigation of the integral versions

$$\mathcal{U}_p(\gamma) := \int_{\mathbb{R}/\mathbb{Z}} \sup_{y,z\in\mathbb{R}/\mathbb{Z}} c^p(\gamma(x),\gamma(y),\gamma(z))|\gamma'(x)|\ dx,$$
$$\mathcal{I}_p(\gamma) := \int_{\mathbb{R}/\mathbb{Z}} \int_{\mathbb{R}/\mathbb{Z}} \sup_{z\in\mathbb{R}/\mathbb{Z}} c^p(\gamma(x),\gamma(y),\gamma(z))|\gamma'(x)||\gamma'(y)|\ \ dx\ dy,$$

and

$$\mathcal{M}_p(\gamma) := \int_{\mathbb{R}/\mathbb{Z}} \int_{\mathbb{R}/\mathbb{Z}} \int_{\mathbb{R}/\mathbb{Z}} c^p(\gamma(x),\gamma(y),\gamma(z))|\gamma'(x)||\gamma'(y)||\gamma'(z)|\ \ dx\ dy\ dz,$$

- a program that was pushed forward in a series of groundbreaking papers by Paweł Strzelecki, Heiko von der Mosel and Marta Szumańska [13, 14] in which they could show, apart from other things, that for suitable p these energies are self repulsive and possess certain regularizing effects - and thus are indeed worth being called "knot energies".

Generalizing the notion of Menger curvature from one-dimensional to higher dimensional objects, was not a trivial task. The obvious generalization – i.e. taking the inverse of the radius of an m-dimensional sphere defined by its $m+2$ points – seems not to be the right Ansatz from an analytic point of view. Strzelecki and von der Mosel have given examples (see [15, Appendix B]) of smooth embedded manifolds for which the resulting integral curvatures are unbounded. But more promising candidates were introduce and successfully investigated in [9, 10, 15] and [6].

In this article we will look at the variant of integral Menger curvature for submanifolds of the Euclidean space of arbitrary dimension and codimension introduced in [6] – and sometimes laxly refer to it as Menger curvature being aware that there are other quantities that deserve this name.

Motivated by the formula

$$c(x,y,z) = 4\frac{\mathcal{H}^2(\Delta(x,y,z))}{|x-y||y-z||z-x|},$$

where $\Delta(x,y,z)$ stands for the convex hull of the points x, y, and z, we are led to use the quantity

$$\mathcal{K}(x_0,\dots,x_{m+1}) = \frac{\mathcal{H}^{m+1}(\Delta(x_0,\dots,x_{m+1}))}{(\operatorname{diam}\{x_0,\dots,x_{m+1}\})^{m+2}}$$

as a substitute for the Menger curvature of curves. Here again $\Delta(x_0,\dots,x_{m+1})$ stands for the convex hull of the points $x_0, \dots, x_{m+1}$ in $\mathbb{R}^n$. We take the diameter of the set of points $\{x_0,\dots,x_{m+1}\}$ to the power $m+2$ in the denominator, which guarantees that this quantity scales like a curvature.

It is easy to check that for triples (x,y,z) we always have $4\mathcal{K}(x,y,z) \le c(x,y,z)$ and that for a class of triangles with comparable sides, (i.e. $|x-y| \simeq |y-z| \simeq |z-x|$) the two quantities $\mathcal{K}(x,y,z)$ and $c(x,y,z)$ are comparable. It is also obvious that for general triangles this is not true.

Following the suggestion of Gonzalez and Maddocks mentioned above, one is led to the following *intermediate integral Menger curvatures*

$$\mathcal{E}_p^k(\Sigma) = \int_{\Sigma^k} \sup_{x_k,\dots x_{m+1}\in\Sigma} \mathcal{K}(x_0,\dots,x_{k-1})^p \; d\mathcal{H}^{mk}_{x_0,\dots,x_{k-1}} .$$

for $k \in \{1,\dots,m+1\}$ and the *integral Menger curvature*

$$\mathcal{E}_p = \mathcal{E}_p^{m+2}(\Sigma) = \int_{\Sigma^{m+2}} \mathcal{K}(x_0,\dots,x_{m+1})^p \; d\mathcal{H}^{m(m+2)}_{x_0,\dots,x_{m+1}}$$

discussed in [6].

The main result of this article is the following characterization of all compact embedded C^1 submanifolds with finite energy $\mathcal{E}_p^k$ for $k \in \{2,\dots,m+1\}$.

Theorem 1.1. *Let $m,n,k \in \mathbb{N}$, $p \in \mathbb{R}$ satisfy $m < n$, $2 \le k \le m+2$ and $p > m(k-1)$. Furthermore, let $\Sigma \subseteq \mathbb{R}^n$ be a compact m-dimensional C^1 manifold and $s = 1 - \frac{m(k-1)}{p} \in (0,1)$. Then $\mathcal{E}_p^k(\Sigma)$ is finite if and only if Σ can locally be represented as the graph of a function belonging to the Sobolev-Slobodeckij space $W^{1+s,p}(\mathbb{R}^m,\mathbb{R}^{n-m})$.*

Here $W^{s,p}$ stands for the Sobolev-Slobodeckij spaces. For a definition of these spaces, some basic properties, and references see Section 2.

Note, that a classification of all finite energy objects for $\mathcal{E}_p^1$ for $p \in [1,\infty]$ was already achieved in [5] and [4] – essentially these are the embedded $W^{2,p}$ submanifolds. So we now have a complete classification of C^1 manifolds with finite energy for all intermediate integral Menger curvatures.

As an immediate consequence of Theorem 1.1 and the results in [6], one gets for the integral Menger curvature

Corollary 1.2. *Let $m,n \in \mathbb{N}$, $p \in \mathbb{R}$ satisfy $m < n$ and $p > m(m+2)$. Furthermore, let Σ be an admissible compact set in the sense of [6]. Then $\mathcal{E}_p(\Sigma)$ is finite if and only if Σ can locally be represented as the graph of some function belonging to the Sobolev-Slobodeckij space $W^{1+s,p}(\mathbb{R}^m,\mathbb{R}^{n-m})$, where $s = 1 - \frac{m(m+1)}{p} \in (0,1)$.*

For curves, the classification of finite energy objects for $\mathcal{M}_p$, $\mathcal{I}_p$ was achieved in [2, 13]. It is a surprising fact, that though $\mathcal{E}_3^p$, $\mathcal{E}_2^p$, $\mathcal{E}_1^p$ for curves look much weaker than $\mathcal{M}_p$, $\mathcal{I}_p$, and $\mathcal{U}_p$, the corresponding energies are bounded on exactly the same objects.

In [7] the optimal Hölder regularity that implies finiteness of $\mathcal{M}_p$ or $\mathcal{E}_p$ was deduced. Note, that this result in any dimension and for all intermediate energies can now be interpreted as a simple consequences of Theorem 1.1 and classical embedding and non-embedding theorems of Sobolev-Slobodeckij spaces.

2 Sobolev-Slobodeckij spaces

For the readers convenience we repeat some well known facts about Sobolev-Slobodeckij spaces.

Definition 2.1 ([1], Chapter VII)**.** Let $k \in \mathbb{N}$, $s \in (0,1)$, $p \ge 1$ and let Ω be an open subset of $\mathbb{R}^m$ with smooth boundary. We say that $u \in L^p(\Omega)$ belongs to the *Sobolev-Slobodeckij space* $W^{k+s,p}(\Omega)$ if

$$\|u\|_{W^{k+s,p}(\Omega)} = \left(\|u\|^p_{W^{k,p}(\Omega)} + \sum_{|\alpha|=k} \int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{|x-y|^{m+sp} \; dy \; dx} \right)^{\frac{1}{p}} < \infty .$$

When we show that boundedness of $\mathcal{E}_p^k$ implies that the submanifold was of class $W^{1+s,p}$, we will use a different but equivalent norm on these spaces due to Besov:

Definition 2.2 ([1], 7.67)**.** Let $k \in \mathbb{N}$, $s \in (0,1)$, $p \geq 1$ and let Ω be an open subset of $\mathbb{R}^m$ with smooth boundary. For $x \in \Omega$, we set

$$\Omega_x = \{y \in \Omega : \tfrac{1}{2}(x+y) \in \Omega\}\,.$$

For $u \in W^{k,p}(\Omega)$ we say that $u \in B^{k+s,p}(\Omega)$ if

$$\|u\|_{B^{k+s,p}(\Omega)} = \left(\|u\|^p_{W^{k,p}(\Omega)} + \sum_{|\alpha|=k} \int_\Omega \int_{\Omega_x} \frac{|D^\alpha u(x) - 2D^\alpha u(\frac{1}{2}(x+y)) + D^\alpha u(y)|^p}{|x-y|^{m+sp}}\, dy\, dx \right)^{\frac{1}{p}}$$

is finite.

Theorem 2.3 (cf. [18], Theorem 2.5.1)**.** *Let $k \in \mathbb{N}$, $s \in (0,1)$, $p \geq 1$ and let Ω be an open subset of $\mathbb{R}^m$ with smooth boundary. Then we have $W^{k+s,p}(\Omega) = B^{k+s,p}(\Omega)$ and the norms $\|\cdot\|_{W^{k+s,p}(\Omega)}$ and $\|\cdot\|_{B^{k+s,p}(\Omega)}$ are equivalent. Moreover, for $\sigma \in (0,2)$ the norm $\|\cdot\|_{B^{\sigma,p}(\Omega)}$ is also equivalent to the following norm*

$$\|u\| = \left(\|u\|^p_{L^p(\Omega)} + \int_\Omega \int_{\Omega_x} \frac{|u(x) - 2u(\frac{1}{2}(x+y)) + u(y)|^p}{|x-y|^{m+\sigma p}}\, dy\, dx \right)^{\frac{1}{p}} < \infty\,.$$

Furthermore, we will use the following extension lemma

Theorem 2.4 (cf. [17], Theorem p. 201)**.** *Let $k \in \mathbb{N}$, $s \in (0,1)$, $p \geq 1$ and let Ω be an open subset of $\mathbb{R}^m$ with smooth boundary. Then $u \in W^{k+s,p}(\Omega)$ if and only if it is the restriction of a function $\tilde{u} \in W^{k+s,p}(\mathbb{R}^m)$ onto Ω.*

Apart from this, we will need the following well known embedding theorem

Theorem 2.5 ([1], Theorem 7.57)**.** *Let $s > 0$ and $1 < p < m$. If $n < (s-j)p$ for some nonnegative integer j, then $W^{s,p}(\Omega) \subset C^j_{loc}(\mathbb{R}^m)$.*

3 Being a $W^{1+s,p}$ submanifold implies $\mathcal{E}^k_p < \infty$

Since we will have to work with balls of different dimensions in this article, let us introduce the symbol $B^l(r,x)$ for the l-dimensional open ball in $\mathbb{R}^l$.

In this section we are proving the following half of our main theorem

Theorem 3.1. *Fix some natural number $2 \leq k \leq m+2$. Let $p > m(k-1)$ and $s = 1 - \frac{m(k-1)}{p} \in (0,1)$. Let $\Sigma \subseteq \mathbb{R}^n$ be a compact m-dimensional manifold, with local graph representation in the Sobolev-Slobodeckij space $W^{1+s,p}(\mathbb{R}^m, \mathbb{R}^{n-m})$. Then $\mathcal{E}^k_p(\Sigma)$ is finite.*

Throughout this section we use the symbol T_k to denote a k-tuple $T = (w_0, \ldots, w_{k-1})$ of k points in $\mathbb{R}^n$. Using this notation we can write

$$\mathcal{K}(T_{m+2}) = \frac{\mathcal{H}^{m+1}(\Delta\, T_{m+2})}{(\operatorname{diam} T_{m+2})^{m+2}}\,.$$

We define the measure

$$\mu_k = \underbrace{\mathcal{H}^m \otimes \cdots \otimes \mathcal{H}^m}_{k \text{ times}}$$

and set

$$\mathcal{K}_k(x_0, \ldots, x_{k-1}) = \sup_{x_k, \ldots, x_{m+1} \in \Sigma} \mathcal{K}(x_0, \ldots, x_{m+1})$$

for $k \in \{1, \ldots, m+1\}$ and

$$\mathcal{K}_{m+2}(x_0, \ldots, x_{k-1}) = \mathcal{K}(x_0, \ldots, x_{m+1})\,.$$

Now, we can write

$$\mathcal{E}_p^k(\Sigma) = \int_{\Sigma^k} \mathcal{K}_k(T_k)^p \; d\mu_k(T_k)$$

for all $k \in \{1, m+2\}$.

For any set $A \subseteq \mathbb{R}^n$ and any $\lambda > 0$ we define

$$\begin{aligned} A^k_{\geq\lambda} &= \{(w_1, \dots, w_k) \in A^k : \operatorname{diam}\{w_1, \dots, w_k\} \geq \lambda\} \\ \text{and} \quad A^k_{<\lambda} &= \{(w_1, \dots, w_k) \in A^k : \operatorname{diam}\{w_1, \dots, w_k\} < \lambda\} = A^k \setminus A^k_{\geq\lambda}\,. \end{aligned}$$

Let $o \in \Sigma$ and let $\rho > \lambda > 0$ be some numbers. We set

$$\Sigma_o^\rho = \Sigma \cap \mathbb{B}^n(o, \rho) \quad \text{and} \quad \mathcal{K}_{k,o,\rho}(w_0, \dots, w_{k-1}) = \sup_{w_k, \dots, w_{m+1} \in \Sigma_o^\rho} \mathcal{K}(w_0, \dots, w_{m+1})$$

and introduce the local version of our energy

$$\mathcal{E}_p^k(\Sigma, \rho, \lambda, o) = \int_{(\Sigma_o^\rho)^k_{<\lambda}} \mathcal{K}_{k,o,2\rho}(T_k)^p \; d\mu_k(T_k)\,.$$

The proof of Theorem 3.1 relies on the following two lemmata, the proof of which we will postpone till the end of this section. The first one tells us, that we only have to consider simplices with small diameter.

Lemma 3.2. *For any $\rho > 0$ there exist $\lambda \in (0, \rho)$, $N \in \mathbb{N}$, an N-tuple of points $x_1, \dots, x_N$ in Σ and a constant $C = C(n,m)$ such that*

$$\mathcal{E}_p^k(\Sigma) \leq C(n,m)\mathcal{H}^m(\Sigma)^k(\lambda^{-p} + \rho^{-p}) + \sum_{i=1}^N \mathcal{E}_p^k(\Sigma, \rho, \lambda, x_i)$$

The second lemma tells us that in order to prove Theorem 3.1 it is enough to get some good estimates for the Jones' β-numbers. Those are given by

Definition 3.3. For $x \in \Sigma$ and $r > 0$ we define the *Jones' β-numbers*

$$\beta(x, r) := \inf \left\{ \frac{\sup_{y \in \Sigma \cap \mathbb{B}^n_(x,r)} \operatorname{dist}(y, H)}{r} : H \text{ an affine } m\text{-dimensional space containing } x \right\}.$$

Lemma 3.4. *There exists a constant $C = C(m,n)$ such that for all $\Sigma \subseteq \mathbb{R}^n$ and $T_{m+2} = (x_0, \dots, x_{m+1}) \in \Sigma^{m+2}$ we have*

$$\mathcal{H}^{m+1}(\Delta\, T_{m+2}) \leq C \beta(x_0, \operatorname{diam}(T_{m+2})) \operatorname{diam}(T_{m+2})^{m+1}$$

and consequently

$$\mathcal{K}(T_{m+2}) \leq C \frac{\beta(x_0, \operatorname{diam}(T_{m+2}))}{\operatorname{diam}(T_{m+2})}\,.$$

In fact, we will only use the following immediate consequence of Lemma 3.4

Corollary 3.5. *For $T_k = (x_0, \dots, x_{k-1}) \in (\Sigma_o^\rho)^k$ we have*

$$\mathcal{K}_{k,o,2\rho}(T_k) \leq C \sup_{x_k, \dots, x_{m+1} \in \Sigma_o^{2\rho}} \frac{\beta(x_0, \operatorname{diam}(x_0, \dots, x_{m+1}))}{\operatorname{diam}(x_0, \dots, x_{m+1})} \leq C \sup_{\operatorname{diam}(T_k) \leq r \leq 4\rho} \frac{\beta(x_0, r)}{r}\,.$$

Let us now show how these lemmata can be used to prove Theorem 3.1:

Proof of Theorem 3.1. Despite the fact that the integrand $\mathcal{K}_k(x_0, \dots, x_{k-1})$ depends on the whole of Σ, Lemma 3.2 tells us that it is enough to show that there is a $\rho > 0$ such $\mathcal{E}_p^k(\Sigma, \lambda, \rho, o)$ is finite for every $\lambda \in (0, \rho)$ and every $o \in \Sigma$. The Sobolev embedding theorem (Theorem 2.5) shows that Σ is a compact C^1 submanifold of $\mathbb{R}^n$. Together with the fact that Σ is locally the graph of a $W^{1+s,p}$ function, this allow us to choose $\rho > 0$ so small that for all $o \in \Sigma$ we have after a suitable rotation of the ambient space

$$(\Sigma - o) \cap \mathbb{B}^n_{10\rho} \subseteq \operatorname{graph}(f) = \big\{(x, f(x)) \in \mathbb{R}^n : x \in \mathbb{R}^m\big\}, \tag{1}$$

for some function $f \in W^{1+s,p}(\mathbb{R}^m, \mathbb{R}^{n-m})$ (depending on the choice of $o \in \Sigma$) that satisfies

$$\forall x, y \in \mathbb{B}^m_{10\rho} \quad |f(x) - f(y)| \le |x - y|. \tag{2}$$

Let $0 \le i < j \le k-1$ and let $u, v \in \Sigma_o^\rho$. We set

$$\begin{aligned} \Sigma_{i,j} &= \big\{(w_0, \dots, w_{k-1}) \in (\Sigma_o^\rho)^k : \operatorname{diam}\{w_0, \dots, w_{k-1}\} = |w_i - w_j|\big\} \\ \text{and} \quad \Sigma(u,v) &= \big\{(w_1, \dots, w_{k-2}) \in (\Sigma_o^\rho)^{k-2} : \operatorname{diam}\{u, w_1, \dots, w_{k-2}, v\} = |v - u|\big\}. \end{aligned}$$

For any $(w_1, \dots, w_{k-2}) \in \Sigma(u,v)$ and $j = 1, \dots, k-2$, we have $|w_j - u| \le |v - u|$. Hence,

$$\mathcal{H}^{m(k-2)}(\Sigma(u,v)) \le \Big(2^m \omega_m |v-u|^m\Big)^{k-2} \le C(m,k)|v-u|^{m(k-2)},$$

where ω_m denotes the volume of the m-dimensional unit ball. Note that

$$(\Sigma_o^\rho)^k = \bigcup \big\{\Sigma_{i,j} : 0 \le i < j \le k-1\big\}.$$

Since $\mathcal{K}$ is invariant under permutations of its parameters, so is $\mathcal{K}_{k,o,2\rho}$ and we have

$$\int_{\Sigma_{i,j}} \mathcal{K}_{k,o,2\rho}(T_k)^p \, d\mu_k(T_k) = \int_{\Sigma_{a,b}} \mathcal{K}_{k,o,2\rho}(T_k)^p \, d\mu_k(T_k),$$

for any $i < j$ and $a < b$. Hence

$$\begin{aligned} \mathcal{E}_p^k(\Sigma, \rho, \lambda, o) &= \int_{(\Sigma_o^\rho)^k_{<\lambda}} \mathcal{K}_{k,o,2\rho}(T_k)^p \, d\mu_k(T_k) \le \sum_{0 \le i < j \le k-1} \int_{\Sigma_{i,j} \cap \{|w_i - w_j| < \lambda\}} \mathcal{K}_{k,o,2\rho}(T_k)^p \, d\mu_k(T_k) \\ &= 2\binom{k}{2} \int_{\Sigma_o^\rho} \int_{\Sigma_o^\rho \cap \mathbb{B}^n(u,\lambda)} \int_{\Sigma(u,v)} \mathcal{K}_{k,o,2\rho}(u, w_1, \dots, w_{k-2}, v)^p \, d\mathcal{H}^{m(k-2)}_{w_1,\dots,w_{k-2}} \, d\mathcal{H}^m_v \, d\mathcal{H}^m_u. \end{aligned} \tag{3}$$

Let $|JF(z)| = \sqrt{\det(DF(z)^* DF(z))}$ denote the Jacobian of $F(z) = (z, f(z))$. Set $\beta_o(x,r) := \beta(o + x, r)$. Using Lemma 3.4 and Corollary 3.5 we may write

$$\begin{aligned} \mathcal{E}_p^k(\Sigma, \rho, \lambda, o) &\le C \int_{\Sigma_o^\rho} \int_{\Sigma_o^\rho \cap \mathbb{B}^n(u,\lambda)} |v-u|^{m(k-2)} \sup_{|u-v| \le r \le 4\rho} \frac{\beta(u,r)^p}{r^p} \, d\mathcal{H}^m_v \, d\mathcal{H}^m_u \\ &\le C \int_{\mathbb{B}^m_\rho} \int_{\mathbb{B}^m(x,\lambda)} |F(y) - F(x)|^{m(k-2)} \sup_{|F(y)-F(x)| \le r \le 4\rho} \frac{\beta_o(F(x),r)^p}{r^p} |JF(x)||JF(y)| \, dy \, dx \\ &\le C' \int_{\mathbb{B}^m_\rho} \int_{\mathbb{B}^m(x,\lambda)} |y-x|^{m(k-2)} \sup_{|y-x| \le r \le 4\rho} \frac{\beta_o(F(x),r)^p}{r^p} \, dy \, dx. \end{aligned} \tag{4}$$

To get to the last line, we used the fact that F satisfies (cf. (2))

$$|y - x| \le |F(y) - F(x)| \le 2|y-x|, \quad \text{hence also} \quad |JF(z)| \le 2^m.$$

We set

$$\Sigma_o^{x,r} = (\Sigma - o) \cap \mathbb{B}^n(F(x), r).$$

Observe that $r\beta(u,r)$ can be estimated by the distance of $\Sigma \cap \mathbb{B}^n(u,r)$ from the affine tangent plane $u + T_u\Sigma$. Hence, recalling the definition of the β-numbers, we get

$$
\begin{aligned}
\beta_o(F(x),r) &\le r^{-1} \inf_{H \in G(n,m)} \sup \left\{ \operatorname{dist}(w, F(x)+H) : w \in \Sigma_o^{x,r} \right\} \\
&\le r^{-1} \sup \left\{ \operatorname{dist}(w, F(x) + T_{F(x)}(\Sigma - o)) : w \in \Sigma_o^{x,r} \right\} \\
&\le r^{-1} \sup \left\{ |F(z) - F(x) - DF(x)(z-x)| : z \in \mathbb{B}^m(x,2r) \right\} \\
&= r^{-1} \sup \left\{ |f(z) - f(x) - Df(x)(z-x)| : z \in \mathbb{B}^m(x,2r) \right\}. \qquad (5)
\end{aligned}
$$

Plugging (5) into (4), we are led to

$$
\mathcal{E}_p^k(\Sigma,\rho,\lambda,o) \le C \int_{\mathbb{B}_\rho^m} \int_{\mathbb{B}^m(x,\lambda)} |y-x|^{m(k-2)} \sup_{\substack{|y-x| \le r \le 4\rho \\ z \in \mathbb{B}^m(x,2r)}} \frac{|f(z)-f(x)-Df(x)(z-x)|^p}{r^{2p}} \, dy \, dx \, .
$$

To estimate the term $|f(z)-f(x)-Df(x)(z-x)|$ we set

$$
g_x(z) = f(z) - f(x) - Df(x)(z-x) \, , \quad \text{then } g_x(x) = 0 \, .
$$

Since $f \in W^{1+s,p} \subseteq W^{1,p}$ and $p > m$, using the Sobolev-Morrey embedding theorem, we obtain

$$
\begin{aligned}
\sup_{z \in \mathbb{B}^m(x,2r)} |g_x(z) - g_x(x)| &\le C \sup_{z \in \mathbb{B}^m(x,2r)} |z-x|^{1-\frac{m}{p}} \left(\int_{\mathbb{B}^m(\frac{1}{2}(z+x), |z-x|)} |Dg_x(t)|^p \, dt \right)^{\frac{1}{p}} \\
&\le \widetilde{C} r^{1-\frac{m}{p}} \left(\int_{\mathbb{B}^m(x,5r)} |Dg_x(t)|^p \, dt \right)^{\frac{1}{p}} = \widehat{C} r^{1-\frac{m}{p}} \left(\int_{\mathbb{B}^m(x,5r)} |Df(t) - Df(x)|^p \, dt \right)^{\frac{1}{p}} ,
\end{aligned}
$$

where the right hand side does not depend on z anymore. Hence, we get the estimate

$$
\mathcal{E}_p^k(\Sigma,\rho,\lambda,o) \le C \int_{\mathbb{B}_\rho^m} \int_{\mathbb{B}^m(x,\lambda)} |y-x|^{m(k-2)} \sup_{r \ge |y-x|} \frac{\left(\int_{\mathbb{B}^m(x,5r)} |Df(t)-Df(x)|^p \, dt \right)}{r^{m+p}} \, dy \, dx \, . \qquad (6)
$$

Observe that

$$
\begin{aligned}
\sup_{r \ge |y-x|} \frac{\int_{\mathbb{B}^m(x,5r)} |Df(t)-Df(x)|^p \, dt}{r^{m+p}} &\le C \sup_{r \ge |y-x|} \int_r^{2r} \frac{\int_{\mathbb{B}^m(x,5r)} |Df(t)-Df(x)|^p \, dt}{\tau^{m+p+1}} \, d\tau \\
&\le C \int_{|y-x|}^{\infty} \frac{\int_{\mathbb{B}^m(x,5\tau)} |Df(t)-Df(x)|^p \, dt}{\tau^{m+p+1}} \, d\tau \\
&\le C \int_{\mathbb{R}^m} \int_{\tau \ge \max\{|t-x|/5, |y-x|\}} \frac{|Df(t)-Df(x)|^p}{\tau^{m+p+1}} \, d\tau \, dt \\
&\le C \int_{\mathbb{R}^m} \frac{|Df(t)-Df(x)|^p}{\max\{|t-x|/5, |y-x|\}^{m+p}} \, dt \, . \qquad (7)
\end{aligned}
$$

Note that for any choice of x, y and t we have

$$
\frac{|y-x|^{m(k-2)}}{\max\{|t-x|, |y-x|\}^{m(k-2)}} \le 1 \, . \qquad (8)
$$

Combining (7) and (8) with (6) and using Fubini's theorem, we are led to

$$
\begin{aligned}
\mathcal{E}_p^k(\Sigma,\rho,\lambda,o) &\le C \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \frac{|Df(t)-Df(x)|^p}{\max\{|t-x|, |y-x|\}^{m+p-m(k-2)}} \, dt \, dy \, dx \\
&\le C \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \frac{|Df(t)-Df(x)|^p}{\max\{|t-x|, |y-x|\}^{m+p-m(k-2)}} \, dy \, dt \, dx \, . \qquad (9)
\end{aligned}
$$

We can compute the innermost integral by dividing it into two parts

$$\int_{\mathbb{R}^m} \frac{|Df(t)-Df(x)|^p}{\max\{|t-x|,|y-x|\}^{m+p-m(k-2)}}\,dy = \int_{|y-x|\le|t-x|} \frac{|Df(t)-Df(x)|^p}{|t-x|^{m+p-m(k-2)}}\,dy$$
$$+\int_{|y-x|>|t-x|} \frac{|Df(t)-Df(x)|^p}{|y-x|^{m+p-m(k-2)}}\,dy = C\frac{|Df(t)-Df(x)|^p}{|t-x|^{p-m(k-2)}}\,. \tag{10}$$

Plugging (10) into (9) we finally get

$$\mathcal{E}_p^k(\Sigma,\rho,\lambda,o) \le C\int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \frac{|Df(t)-Df(x)|^p}{|t-x|^{p-m(k-2)}}\,dt\,dx \le C\|Df\|_{W^{s,p}}\,,$$

where $s = 1-\frac{m(k-1)}{p}$. □

Proof of Lemma 3.4. For a linear subspace W of $\mathbb{R}^n$ let P_W denote the orthogonal projection of $\mathbb{R}^n$ onto W and $P_W^\perp := id_{\mathbb{R}^n} - P_W$ be the orthogonal projection of $\mathbb{R}^n$ onto the orthogonal complement of V. Furthermore, let $\mathbf{T} = \Delta T_{m+2}$ and let $d = \operatorname{diam}(\mathbf{T})$.

Without loss of generality we can assume that $x_0 = 0$. If the vectors $\{x_1,\dots,x_{m+1}\}$ are not linearly independent, then $\mathcal{H}^{m+1}(\mathbf{T}) = 0$ and the statement of the lemma is true.

Let $x_1,\dots x_{m+1}$ be linearly independent and let W denote the $(m+1)$-dimensional vector space spanned be these vectors. Set

$$\mathbf{S} := \{s\in W^\perp : |s| \le \beta(0,d)d\}\,.$$

Then, for the set $\mathbf{T}+\mathbf{S}$, using Fubini's theorem we obtain

$$\mathcal{H}^n(\mathbf{T}+\mathbf{S}) = \mathcal{H}^{m+1}(\mathbf{T})\mathcal{H}^{n-m-1}(\mathbf{S}) = \omega_m\mathcal{H}^{m+1}(\mathbf{T})d^{n-m-1}\beta(0,d)^{n-m-1} \tag{11}$$

where ω_m is the volume of the m-dimensional unit ball.

From the definition of the Jones' β-numbers we can find a sequence of m-dimensional vector spaces V_j such that

$$\sup_{y\in\Sigma\cap\mathbb{B}^n(x_0,d)} |P_{V_j}^\perp(y)| \le \big(\beta(x_0,d)+\tfrac{1}{j}\big)d\,.$$

Since the Grassmannian $G(n,m)$ of all m-dimensional subspaces of $\mathbb{R}^n$ is a compact manifold, we can find a subsequence V_{j_k} converging to some $V\in G(n,m)$. Observe also that the mapping $Q: G(n,m)\to\mathbb{R}^n$ given by $Q(V) = P_V(y)$ is continuous[1] for any choice of $y\in\mathbb{R}^n$. In consequence, we get the estimate

$$\forall y\in\Sigma\cap\mathbb{B}^n(x_0,d) \quad |P_V^\perp(y)| \le \beta(x_0,d)d\,.$$

The vertices of $\mathbf{T}$ lie in $\Sigma\cap\mathbb{B}^n(x_0,d)$ and $\mathbf{T}$ is convex, so we also have

$$\forall t\in\mathbf{T} \quad |P_V^\perp(t)| \le \beta(x_0,d)d\,.$$

Let $y\in\mathbf{T}+\mathbf{S}$ and let $t\in\mathbf{T}$ and $s\in\mathbf{S}$ be such that $s+t=y$. Using the triangle inequality we see that

$$|P_V(y)| \le |y| \le (1+\beta(0,d))d$$

and

$$|P_V^\perp(y)| \le |P_V^\perp(t)| + |P_V^\perp(s)| \le 2\beta(x_0,d)d\,.$$

Hence, $\mathbf{T}+\mathbf{S}$ is a subset of

$$Z = \big\{y\in\mathbb{R}^n : |P_V(y)|\le 2d,\ |P_V^\perp(y)| \le 2\beta(0,d)d\big\}\,.$$

Using again Fubini's theorem, we obtain

$$\mathcal{H}^n(\mathbf{T}+\mathbf{S}) \le \mathcal{H}^n(Z) = C2^{n-m}\beta(0,d)^{n-m}d^n\,. \tag{12}$$

[1] The metric on $G(n,m)$ is defined by the formula $\operatorname{dist}(U,V) = \|P_U - P_V\|$.

Combining (11) and (12) we finally deduce

$$\mathcal{H}^{m+1}(T) \le C\beta(0,d)d^{m+1} .$$

□

Proof of Lemma 3.2. Fix some $\rho > 0$. Since Σ is compact, we can cover it by a finite number of balls of radius ρ

$$\Sigma \subseteq \bigcup_{i=1}^{N} \mathbb{B}^n(x_i,\rho)\,, \quad \text{where } x_i \in \Sigma \text{ for } i = 1,\dots,N\,.$$

This covering has its Lebesgue number, say $\lambda \in (0,\rho)$, so that any set of points in Σ of diameter less than λ lies entirely in one of the balls $\mathbb{B}^n(x_i,\rho)$ for some $i \in \{1,\dots,N\}$. Observe that if the diameter $\operatorname{diam}(T_k) \ge \lambda$, then $\mathcal{K}_k(T_k) \le C(n,m)\lambda^{-1}$. Also, if $w_0,\dots,w_{k-1} \in \Sigma^{\rho}_{x_i}$ and $w_k,\dots,w_{m+1} \in \Sigma \setminus \Sigma^{2\rho}_{x_i}$, then the diameter $\operatorname{diam}(w_0,\dots,w_{m+1}) \ge \rho$ and we have $\mathcal{K}(w_0,\dots,w_{m+1}) \le C(n,m)\rho^{-1}$. Hence, for $T_k \in (\Sigma^{\rho}_{x_i})^k$, we have

$$\begin{aligned}\mathcal{K}_k(T_k) \le &\sup_{w_k,\dots,w_{m+1}\in\Sigma^{2\rho}_{x_i}} \mathcal{K}(w_0,\dots,w_{m+1}) \\ &\qquad + \sup_{w_k,\dots,w_{m+1}\in\Sigma\setminus\Sigma^{2\rho}_{x_i}} \mathcal{K}(w_0,\dots,w_{m+1}) \le \mathcal{K}_{k,x_i,2\rho}(T_k) + C(n,m)\rho^{-1}\,.\end{aligned}$$

In consequence, the following estimate holds

$$\begin{aligned}\mathcal{E}^k_p(\Sigma) = &\int_{\Sigma^k_{\ge\lambda}} \mathcal{K}_k(T_k)^p \; d\mu_k(T_k) + \int_{\Sigma^k_{<\lambda}} \mathcal{K}_k(T_k)^p \; d\mu_k(T_k) \\ &\le \widetilde{C}(n,m)\mathcal{H}^m(\Sigma)^k(\lambda^{-p}+\rho^{-p}) + \sum_{i=1}^{N} \int_{(\Sigma^{\rho}_{x_i})^k_{<\lambda}} \mathcal{K}_{k,x_i,2\rho}(T_k)^p \; d\mu_k(T_k)\,.\end{aligned}$$

□

4 Regularizing effects of $\mathcal{E}^k_p$

Let us now prove the other implication of the main theorem, i.e.

Theorem 4.1. *Let $k \in \{2,\dots,m+2\}$ and $p > m(k-1)$ and Σ be an m-dimensional embedded C^1 submanifold of the Euclidean space $\mathbb{R}^n$. If $\mathcal{E}^k_p(\Sigma)$ is finite, then Σ is locally given by graphs functions in $W^{1+s,p}(\mathbb{R}^m,\mathbb{R}^{n-m})$, where $s = 1 - \frac{m(k-1)}{p}$.*

Before we start, let us recall a definition of the outer product:

Definition 4.2. Let $w_1, \dots, w_l$ be some vectors in $\mathbb{R}^n$. We define *the outer product* $w_1 \wedge\dots\wedge w_l$ to be a vector in $\mathbb{R}^{\binom{n}{l}}$, whose coordinates are exactly the l-minors of the $(l \times n)$-matrix $(w_1,\dots,w_l)$. The coordinates of $w_1 \wedge\dots\wedge w_l$ are indexed by l-tuples $(i_1,\dots,i_l)$, where $i_j \in \{1,\dots,n\}$ for each $j = 1,\dots,l$ and $i_1 < i_2 < \dots < i_l$.

Remark 4.3. A standard fact from linear algebra says that the length $|w_1 \wedge\dots\wedge w_l|$ of an outer product of $w_1, \dots, w_l$ is equal to the l-dimensional volume of the parallelotope spanned by $w_1, \dots, w_l$.

Now we are ready to prove Theorem 4.1:

Proof of Theorem 4.1. For a point $p \in \Sigma$ we have to show that a small neighborhood of p in Σ can be given as the graph of a $W^{1+s,p}$ function on $\mathbb{R}^m$. We can assume after a suitable translation

and rotation that $p = 0$ and since Σ is of class C^1 that there is a function $f \in C^1(\mathbb{R}^m, \mathbb{R}^{n-m})$ satisfying $f(0) = 0$,

$$\|Df\|_{L^\infty} \leq 1 \quad \text{and} \quad g(\mathbb{B}^m_{2\delta}) \subset \Sigma,$$

where $g(x) = (x, f(x))$. We will show that then $f \in B^{1+s,p}(\mathbb{B}^m_\delta)$. Using that $B^{1+s,p}(\mathbb{B}^m_\delta) = W^{1+s,p}(\mathbb{B}^m_\delta)$ by Theorem 2.3 and the extension Theorem 2.4 this proves Theorem 4.1.

Recalling Definition 4.2 and Remark 4.3, for $y, w_1, \ldots, w_{m+1} \in \mathbb{B}^m_\delta$ the following holds

$$\begin{aligned}
&\mathcal{H}^{m+1}(\Delta(g(y), g(y+w_1), \ldots, g(y+w_{m+1}))) \\
&\qquad = \frac{1}{m+1} |(g(y+w_1) - g(y)) \wedge \ldots \wedge (g(y+w_{m+1}) - g(y))| \\
&\qquad\qquad = \frac{1}{m+1} \left| \begin{pmatrix} f(y+w_1) - f(y) \\ w_1 \end{pmatrix} \wedge \ldots \wedge \begin{pmatrix} f(y+w_{m+1}) - f(y) \\ w_{m+1} \end{pmatrix} \right| . \qquad (13)
\end{aligned}$$

For fixed $w_1 \in \mathbb{R}^n$ let us set

$$\begin{aligned}
\Omega^k_{w_1} := \Big\{ (w_2, \ldots, w_{k-1}) \in (\mathbb{B}^m_\delta)^{k-2}) : &|w_i| \leq |w_1| \ \forall i \in \{2, \ldots, k-1\}, \\
&|w_2 \wedge \ldots \wedge w_{k-1}| \geq \frac{1}{2} |w_1|^{k-2} \Big\} .
\end{aligned}$$

An easy scaling argument leads to

$$\mathcal{H}^{m(k-2)}(\Omega^k_{w_1}) = |w_1|^{m(k-2)} \mathcal{H}^{m(k-2)}(\Omega^k_{\frac{w_1}{|w_1|}}) = c|w_1|^{m(k-2)} , \quad \text{where } c = \mathcal{H}^{m(k-2)}(\Omega^k_{\frac{w_1}{|w_1|}}) \qquad (14)$$

obviously does not depend on w_1.

Remark 4.4. Please note that all the following estimates also hold for $k = m + 2$ using the convention that there is no supremum and $m + 2$ integrals in this case.

Using (13) we can write

$$\begin{aligned}
\mathcal{E}^k_p(\Sigma) &= \int_{\Sigma^k} \sup_{x_k, \ldots, x_{m+1} \in \Sigma} \mathcal{K}(x_0, \ldots, x_{m+1})^p \, d\mathcal{H}^{mk}_{x_0, \ldots, x_{k-1}} \\
&\geq c \int_{(\mathbb{B}^m_\delta)^k} \sup_{\substack{w_j \in \mathbb{B}^m_\delta \\ j=k,\ldots,m+1}} \frac{\mathcal{H}^{m+1}(\Delta(g(y), g(y+w_1), \ldots, g(y+w_{m+1})))^p}{\operatorname{diam}(T)^{p(m+2)}} \, dw_{k-1} \cdots dw_1 \, dy \\
&\geq \bar{c} \int_{(\mathbb{B}^m_\delta)^2} \int_{\Omega^k_{w_1}} |w_1|^{-p(m+2)} \sup_{\substack{w_j \in \mathbb{B}^m_\delta \\ j=k,\ldots,m+1}} \left| \begin{pmatrix} f(y+w_1) - f(y) \\ w_1 \end{pmatrix} \wedge \cdots \wedge \begin{pmatrix} f(y+w_{m+1}) - f(y) \\ w_{m+1} \end{pmatrix} \right|^p \\
&\qquad\qquad dw_{k-1} \cdots dw_1 \, dy .
\end{aligned}$$

Now, we use a simple trick: we write the last line as $\tilde{c}/2$ times twice the integral. We leave the first as it is and substitute $w_1 \mapsto -w_1$ in the second integral to get

$$\begin{aligned}
\mathcal{E}^k_p(\Sigma) \geq \frac{\bar{c}}{2} \Bigg\{ &\int_{(\mathbb{B}^m_\delta)^2} \int_{\Omega^k_{w_1}} |w_1|^{-p(m+2)} \sup_{\substack{w_j \in \mathbb{B}^m_\delta \\ j=k,\ldots,m+1}} \left| \begin{pmatrix} f(y+w_1) - f(y) \\ w_1 \end{pmatrix} \wedge \cdots \right. \\
&\qquad \left. \cdots \wedge \begin{pmatrix} f(y+w_{m+1}) - f(y) \\ w_{m+1} \end{pmatrix} \right|^p \, dw_{k-1} \cdots dw_1 \, dy \\
&+ \int_{(\mathbb{B}^m_\delta)^2} \int_{\Omega^k_{w_1}} |w_1|^{-p(m+2)} \sup_{\substack{w_j \in \mathbb{B}^m_\delta \\ j=k,\ldots,m+1}} \left| \begin{pmatrix} f(y-w_1) - f(y) \\ -w_1 \end{pmatrix} \wedge \cdots \right. \\
&\qquad\qquad \left. \cdots \wedge \begin{pmatrix} f(y+w_{m+1}) - f(y) \\ w_{m+1} \end{pmatrix} \right|^p \, dw_{k-1} \cdots dw_1 \, dy \Bigg\} .
\end{aligned}$$

Next, we apply the triangle inequality for the supremum norm obtaining

$$\mathcal{E}_p^k(\Sigma) \geq \frac{\bar{c}}{2} \int_{(\mathbb{B}_\delta^m)^2} \int_{\Omega_{w_1}^k} |w_1|^{-p(m+2)} \sup_{\substack{w_j \in \mathbb{B}_\delta^m \\ j=k,\dots,m+1}} \left| \begin{pmatrix} f(y+w_1) - 2f(y) + f(y-w_1) \\ 0 \end{pmatrix} \wedge \cdots \right.$$
$$\left. \cdots \wedge \begin{pmatrix} f(y+w_{m+1}) - f(y) \\ w_{m+1} \end{pmatrix} \right|^p \, dw_{k-1} \cdots dw_1 \, dy \,. \quad (15)$$

To estimate this further, for a given $w_1 \in \mathbb{R}^n$ and $(w_2, \dots, w_{k-1}) \in \Omega_{w_1}^k$, we choose vectors $w_k, \dots, w_{m+1}$ such that $w_k/|w_1|, \dots, w_{m+1}/|w_1|$ forms an orthonormal basis of the orthogonal complement of $\operatorname{span}(w_2, \dots, w_{k-1})$. For $k = 1, \dots, n$. Furthermore, we let $e \in \mathbb{R}^{n-m}$ be a unit vector satisfying $\langle f(y+w_1) - 2f(y) + f(y-w_1), e\rangle = |f(y+w_1) - 2f(y) + f(y-w_1)|$ and we set $X = \operatorname{span}\{(e,0), (0,w_2), \dots, (0,w_{m+1})\} \subseteq \mathbb{R}^n$. For brevity of notation we set

$$v = f(y+w_1) - 2f(y) + f(y-w_1) \in \mathbb{R}^{n-m} \,.$$

Observe that the orthogonal projection onto X cannot increase the $(m+1)$-dimensional measure of any set. Employing the fact that $(e,0)$ is orthogonal to each of $(0,w_k)$ for $k = 2, \dots, m+1$ and then using Laplace expansion of the determinant with respect to the first column, we obtain[2]

$$\begin{aligned}
&\left| \begin{pmatrix} v \\ 0 \end{pmatrix} \wedge \begin{pmatrix} f(y+w_2) - f(y) \\ 0 \end{pmatrix} \wedge \ldots \wedge \begin{pmatrix} f(y+w_{m+1}) - f(y) \\ w_{m+1} \end{pmatrix} \right| \\
&\geq \left| \begin{pmatrix} \langle v, e\rangle \\ 0 \end{pmatrix} \wedge \begin{pmatrix} \langle f(y+w_2) - f(y), e\rangle \\ w_2 \end{pmatrix} \wedge \ldots \wedge \begin{pmatrix} \langle f(y+w_{m+1}) - f(y), e\rangle \\ w_{m+1} \end{pmatrix} \right| \\
&= \left| \det \begin{pmatrix} \langle v, e\rangle & \langle f(y+w_2) - f(y), e\rangle & \cdots & \langle f(y+w_{m+1}) - f(y), e\rangle \\ 0 & w_2 & \cdots & w_{m+1} \end{pmatrix} \right| \\
&= |f(y+w_1) - 2f(y) + f(y-w_1)| |w_2 \wedge \ldots \wedge w_{k-1}| |w_1|^{m+2-k} \\
&\geq \frac{1}{2} |f(y+w_1) - 2f(y) + f(y-w_1)| |w_1|^m \,.
\end{aligned}$$

Hence, for all $w_1 \in \mathbb{R}^m$ and $(w_2, \dots, w_{k-1}) \in \Omega_{w_1}^k$ we have

$$\sup_{w_k, \dots, w_{m+1} \in \mathbb{B}_{|w_1|}^m} \left| \begin{pmatrix} f(y+w_1) - 2f(y) + f(y-w_1) \\ 0 \end{pmatrix} \wedge \cdots \wedge \begin{pmatrix} f(y+w_{m+1}) - f(y) \\ w_{m+1} \end{pmatrix} \right|$$
$$\geq \frac{1}{2} |f(y+w_1) - 2f(y) + f(y-w_1)| |w_1|^m \,. \quad (16)$$

Plugging (16) into (15), we finally get

$$\begin{aligned}
\mathcal{E}_p^k(\Sigma) \geq c \int_{(\mathbb{B}_\delta^m)^2} \int_{\Omega_{w_1}^k} |w_1|^{-p(m+2)} &\sup_{w_k, \dots, w_{m+1} \in \mathbb{B}_{|w_1|}^m} \left| \begin{pmatrix} f(y+w_1) - 2f(y) + f(y-w_1) \\ 0 \end{pmatrix} \wedge \cdots \right. \\
&\left. \cdots \wedge \begin{pmatrix} f(y+w_{m+1}) - f(y) \\ w_{m+1} \end{pmatrix} \right|^p \, dw_{k-1} \cdots dw_1 \, dy \\
&\geq \bar{c} \int_{(\mathbb{B}_\delta^m)^2} \int_{\Omega_{w_1}^k} \frac{|f(y+w_1) - 2f(y) + f(y-w_1)|^p}{|w_1|^{p(m+2)-pm}} \, dw_{k-1} \cdots dw_1 \, dy \\
&= \tilde{c} \int_{(\mathbb{B}_\delta^m)^2} \frac{|f(y+w_1) - 2f(y) + f(y-w_1)|^p}{|w_1|^{2p-m(k-2)}} \, dw_1 \, dy \,.
\end{aligned}$$

By Theorem 2.3, we thus have $f \in W^{\tilde{s},p}(\mathbb{B}_\delta^m)$, where $\tilde{s}$ is given through the relation $m + \tilde{s}p = 2p - m(k-2)$ and hence $\tilde{s} = 2 - \frac{m(k-1)}{p} = 1 + s$. □

[2]Note that in the first line the wedged vectors are n-dimensional while in the second line they are $(m+1)$-dimensional.

Acknowledgements

The first author was supported by Swiss National Science Foundation Grant Nr. 200020 125127 and "The Leverhulme Trust".
The second author was supported by the Polish Ministry of Science grant no. N N201 397737 (years 2009-2012).

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